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# On the continuous limit of integrable lattices III. Kupershmidt systems and $sl(N+1)$ KdV theories

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**Abstract.** We discuss the connection between the zero-spacing limit of the $N$ fields Kupershmidt lattice and the KdV-type theory corresponding to the Lie algebra $sl(N+1)$. The case $N=2$ is worked out in detail, recovering from the limit process the Boussinesq theory with its infinitely many commuting vector fields, their Lax pairs and Hamiltonian formulations. The 'recombination method' proposed here to derive the Boussinesq hierarchy from the limit of the $N=2$ Kupershmidt system works, in principle, for arbitrary $N$.

## 1. Introduction

In this paper we conclude an analysis of the relations between certain integrable lattices and the KdV-type theories.

In [1] we have considered a class of lattices, referred to as *Volterra systems*; in the 'interpolated version', which is the starting point for the continuous limit, the phase space of the $N$-fields Volterra ($V_N$) system is a set of $N$-tuples $\mathcal{A}=(a_1,\ldots,a_N)$, where each element $a_p$ is a real smooth function of a continuous variable $x$. The Lax operator is

$$L^{\epsilon}(\mathcal{A})=\frac{1}{2}\sum_{p=1}^{N}(a_p\Delta_{(2p-1)\epsilon}+a_{p[(-2p+1)\epsilon]}\Delta_{(-2p+1)\epsilon}). \tag{1.1}$$

Here, for each displacement $\eta$ and each function $f$ of $x$, we indicate with $f_{[\eta]}$ the shifted function $x\mapsto f(x+\eta)$ (so the notation $a_{p[\eta]}$ stands for the function $x\mapsto a_p(x+\eta)$). Also, we denote with $\Delta_\eta$ the shift operator sending any function $f$ into $f_{[\eta]}$. All the shifts $\eta$ considered in equation (1.1) are integer multiples of a fundamental shift $\epsilon$, representing the lattice spacing.

The limit $\epsilon\mapsto 0$, in which $(\Delta_\epsilon-1)/\epsilon\mapsto\partial_x$, has been analysed in [1]; in this way, a strict relation has been pointed out between the $V_N$ system and a KdV-type theory in $N$ fields. This limiting KdV theory in the fields $\boldsymbol{u}=(u_1,\ldots,u_N)$ is associated (in the Drinfeld–Sokolov sense [2]) to the Lie algebra $sp(N)$, and rests on the Lax operator

$$L^{sp(N)}(\boldsymbol{u}):=\partial^{2N}+\frac{1}{2}\sum_{l=1}^{N}(u_l\partial^{2N-2l}+\partial^{2N-2l}u_l). \tag{1.2}$$

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In the case $N = 2$, the main structures describing the integrability of the $sp(2)$ KdV (Lax pairs, Hamiltonian structure, hierarchies of conserved functionals and commuting vector fields) have been fully reconstructed from the $\epsilon \mapsto 0$ limit of homologous $V_2$ objects. The limit process for the case $N = 1$ had also been analysed in [3]; the $V_1$ theory is well known in the literature on integrable lattices as the Kac–Moerbeke system [4], and its zero-spacing limit is the ordinary KdV theory.

In the present paper we analyse along the same lines another class of integrable lattices, introduced in [5], to which we refer as the $N$-fields Kupershmidt (or $K_N$) systems ($N = 1, 2, 3, \ldots$). Working directly in the interpolated version, we can describe the $K_N$ phase space as a set of $N$-tuples $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$; each $\alpha_k$ is a real smooth function of the variable $x$, ranging over the torus $\boldsymbol{T} := \boldsymbol{R}/\boldsymbol{Z}$. The Lax operator, depending on the lattice spacing $\epsilon$, is

$$L^\epsilon(\boldsymbol{\alpha}) = \Delta_\epsilon + \sum_{k=1}^{N} \alpha_{2k-1} \Delta_{(-2k+1)\epsilon}. \tag{1.3}$$

This operator gives rise to a hierarchy of infinitely many commuting vector fields, which are Hamiltonian with respect to a quadratic Poisson tensor (this could be interpreted as the reduction of a conveniently defined quadratic $R$-matrix Poisson tensor on the algebra of differential-difference operators).

If $N = 1$, the corresponding theory is equivalent to that based on the symmetric operator (1.1); in our terminology, the $K_1$ and $V_1$ systems are two isomorphic realizations of the Kac–Moerbeke theory. In contrast, the $V_N$ and $K_N$ systems are essentially different for $N > 1$.

Such a difference can also be traced back by considering the $\epsilon \mapsto 0$ limit. It was proved in [5] that the limit of the $K_N$ Lax operator under a suitable field rescaling $\boldsymbol{\alpha} \mapsto \boldsymbol{u} = (u_1, \ldots, u_N)$ is

$$L(\boldsymbol{u}) = \partial_x{}^{N+1} + \sum_{k=1}^{N} u_k \partial_x{}^{N-k}. \tag{1.4}$$

This is the Lax operator of the KdV-type (or Gelfand–Dickey) theory corresponding to the Lie algebra $sl(N + 1)$, namely the ordinary KdV for $N = 1$, the Boussinesq theory for $N = 2$, etc.

In spite of these results on the limit of the $K_N$ Lax operator (1.3), it is still worth discussing the $\epsilon \mapsto 0$ procedure, so as to reconstruct the full structure characterizing the integrability of the limiting KdV-type theory. In particular, it is of interest to recover its infinitely many commuting vector fields (a topic not discussed in [5]) with the associated Lax pairs and Hamiltonian formulations.

In this paper we illustrate a method to carry on this program. The proposed approach will be worked in detail for the Boussinesq case $N = 2$; the case $N = 1$ will be employed to introduce the main technique, and the case $N = 3$ will also be considered in relation with the Poisson tensor. A distinguished feature of these constructions is the necessity to recombine linearly the vector fields of the hierarchy, the Hamiltonian functions and the companions of the Lax operator in order to recover the corresponding KdV-type structures for $\epsilon \mapsto 0$; the constant coefficients appearing in these linear combinations are determined algorithmically.

The algorithm employed here is different from the recursive one proposed in [6] for recombining the vector fields in the $N = 1$ case (and independently developed in [3], starting from the $V_1$ Lax operator). This iterative method is based on the $\epsilon \mapsto 0$ limit of the biHamiltonian recursion relations; its extension from the $K_1$ to the general $K_N$ system is not possible, because only one (local) Hamiltonian structure is known for $N > 1$.

To overcome this difficulty, we will give alternative characterizations of the recombination coefficients, named $C_s$ conditions (for better clarity, $C_s^{(1)}$ for the $N = 1$ theory, $C_s^{(2)}$ for $N = 2$, etc); these conditions rest on the $\epsilon$-expansion of the Kupershmidt Lax pairs and Hamiltonian functions at a particular point of the phase space. A compatibility argument allows us to characterize the global behaviour of the recombined objects on the grounds of the $C_s$ prescriptions at the particular point; in this way the recombination problem, living in principle in a functional space, is reduced to the solution of a linear system in a finite number of numerical unknowns. A similar idea was employed in [1] for the $V_N$ systems.

The paper is organized as follows. In section 2 we use the $K_1$ (or Kac–Moerbeke) system to exemplify the $C_s^{(1)}$ method. In section 3 we work in detail the $K_2$ system and recover the Boussinesq theory in the $\epsilon \mapsto 0$ limit; the main theorem on the recombination scheme (proposition 3.3), which justifies the $C_s^{(2)}$ algorithm, is proved in section 4. In section 5 we briefly discuss the $K_N$ system for $N$ arbitrary, and show by direct computation that the $N = 3$ Poisson tensor gives, in the $\epsilon \mapsto 0$ limit, the Poisson structure of the $sl(4)$ KdV-type theory. Even though the framework in which we work is well defined on purely theoretical grounds, the explicit constructions require considerable effort from the computational viewpoint; therefore, many have been realized using the *Mathematica* package.

## 2. The $K_1$ system and its continuous limit

Our description of this system and its continuous limit will be concise, because this topic has already been considered in the two known formulations, both with a Kupershmidt-type Lax operator [6] and with a symmetric one [1, 3]. From the viewpoint of the present work, this system is the $N = 1$ case of the Kupershmidt $N$-fields theory; we briefly discuss it here to make the paper self-contained. In comparison with [6], we add some facts concerning the continuous limit of the Hamiltonians and the companions of the Lax operator.

The Lax pairs and the Hamiltonian formulation of the $K_1$ hierarchy are summarized in table 1. We denote with $\mathcal{A}$ the phase space of the system, each point of which is a smooth function $\alpha = \alpha(x)$.

In the definition of $A_s^\epsilon$ given in the table, the subscript $+$ denotes the projection on the shift operators of non-negative order; the trace tr of a differential-shift operator is the integral over $x$ of the zero-order term in the shift†; this notation will also be employed in the rest of the paper.

The Poisson tensor, the vector fields and the Hamiltonians reported in table 1 coincide *exactly* (also in the normalizations) with the homologous objects of [1]; these were expressed in terms of another field variable $a$, related to the present one by $\alpha = \frac{1}{4}a^2_{[-\epsilon]}$. With appropriate specifications, one could show that this transformation sets up a gauge equivalence between the symmetric Lax operator considered in [1] and the Kupershmidt-type operator employed here.

As for the KdV theory, we adopt all the notation and normalization conventions

† More precisely, consider an operator $G := \sum_p g_p \Delta_{p\epsilon}$, where the index $p$ ranges over the relative integers and each coefficient is a smooth function $g_p(x)$. Then, by definition,

$$G_+ := \sum_{p\geqslant 0} g_p \Delta_{p\epsilon} \qquad \mathrm{tr}\, G := \int \mathrm{d}x\, g_0(x).$$

**Table 1.** The $K_1$ (or Kac–Moerbeke) system.

Lax formulation for the $s$th vector field $X_s^\epsilon$ : $\mathrm{d}L^\epsilon/\mathrm{d}t_s = [A_s^\epsilon, L^\epsilon]$

$L^\epsilon(\alpha) := \Delta_\epsilon + \alpha\Delta_{-\epsilon}$ $\quad A_s^\epsilon(\alpha) := \frac{1}{2}(L^\epsilon)_+^{2s}(\alpha)$ $\quad (s = 0, 1, 2, \ldots.)$.

Poisson tensor at a point $\alpha$ : $Q_\alpha^\epsilon : T_\alpha^*\mathcal{A} \to T_\alpha\mathcal{A}$

$Q_\alpha^\epsilon = \frac{1}{2}(\alpha\alpha_{[\epsilon]}\Delta_\epsilon - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon})$.

Hamiltonian formulation: $X_s^\epsilon = Q^\epsilon\,\mathrm{d}f_s^\epsilon$ $\quad (s = 0, 1, 2, \ldots)$.

Hamiltonian functions:

$$f_0^\epsilon(\alpha) := \frac{1}{2}\int \mathrm{d}x \log\alpha \qquad f_s^\epsilon(\alpha) := \frac{1}{2s}\,\mathrm{tr}(L^\epsilon)^{2s}(\alpha) \qquad (s = 1, 2, 3, \ldots).$$

First companions of the Lax operator:

$$A_0^\epsilon(\alpha) = \tfrac{1}{2}$$

$$A_1^\epsilon(\alpha) = \tfrac{1}{2}\Delta_{2\epsilon} + \tfrac{1}{2}(\alpha + \alpha_{[\epsilon]})$$

$$A_2^\epsilon(\alpha) = \tfrac{1}{2}\Delta_{4\epsilon} + \tfrac{1}{2}(\alpha + \alpha_{[2\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{2\epsilon} + \tfrac{1}{2}(\alpha^2 + \alpha\alpha_{[-\epsilon]} + 2\alpha\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[\epsilon]}^2)$$

$$\begin{aligned} A_3^\epsilon(\alpha) := {} & \tfrac{1}{2}\Delta_{6\epsilon} + \tfrac{1}{2}(\alpha + \alpha_{[5\epsilon]} + \alpha_{[2\epsilon]} + \alpha_{[4\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{4\epsilon} \\ & + \tfrac{1}{2}(\alpha^2 + \alpha\alpha_{[2\epsilon]} + \alpha_{[2\epsilon]}^2 + \alpha\alpha_{[-\epsilon]} + \alpha\alpha_{[3\epsilon]} + 2\alpha_{[2\epsilon]}\alpha_{[3\epsilon]} \\ & + \alpha_{[4\epsilon]}\alpha_{[3\epsilon]} + \alpha_{[3\epsilon]}^2 + 2\alpha\alpha_{[\epsilon]} + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[3\epsilon]}\alpha_{[\epsilon]} + \alpha_{[\epsilon]}^2)\Delta_{2\epsilon} \\ & + \tfrac{1}{2}(\alpha^3 + 2\alpha^2\alpha_{[-\epsilon]} + \alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} + \alpha\alpha_{[-\epsilon]}^2 + 3\alpha^2\alpha_{[\epsilon]} + 2\alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} \\ & + \alpha_{[2\epsilon]}^2\alpha_{[\epsilon]} + 2\alpha\alpha_{[-\epsilon]}\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]} + 3\alpha\alpha_{[\epsilon]}^2 + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}^2 + \alpha_{[\epsilon]}^3). \end{aligned}$$

First Hamiltonians after $f_0^\epsilon$:

$$f_1^\epsilon(\alpha) = \int \mathrm{d}x\,\alpha$$

$$f_2^\epsilon(\alpha) = \frac{1}{2}\int \mathrm{d}x(\alpha^2 + 2\alpha\alpha_{[\epsilon]})$$

$$f_3^\epsilon(\alpha) = \frac{1}{3}\int \mathrm{d}x(\alpha^3 + 3\alpha^2\alpha_{[\epsilon]} + 3\alpha\alpha_{[\epsilon]}^2 + 3\alpha\alpha_{[\epsilon]}\alpha_{[2\epsilon]}).$$

First vector fields:

$$X_0^\epsilon(\alpha) = 0$$

$$X_1^\epsilon(\alpha) = \tfrac{1}{2}\alpha(\alpha_{[\epsilon]} - \alpha_{[-\epsilon]})$$

$$X_2^\epsilon(\alpha) = \tfrac{1}{2}\alpha(-\alpha\alpha_{[-\epsilon]} - \alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} - \alpha_{[-\epsilon]}^2 + \alpha\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[\epsilon]}^2)$$

$$\begin{aligned} X_3^\epsilon(\alpha) = {} & \tfrac{1}{2}\alpha(-\alpha^2\alpha_{[-\epsilon]} - \alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} - \alpha_{[-2\epsilon]}^2\alpha_{[-\epsilon]} - \alpha_{[-2\epsilon]}\alpha_{[-3\epsilon]}\alpha_{[-\epsilon]} - 2\alpha\alpha_{[-\epsilon]}^2 - 2\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}^2 - \alpha_{[-\epsilon]}^3 \\ & + \alpha^2\alpha_{[\epsilon]} + \alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}^2\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]} + 2\alpha\alpha_{[\epsilon]}^2 + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]}^2 + \alpha_{[\epsilon]}^3). \end{aligned}$$

of [1, 3]. In particular $\mathcal{U}$ denotes the phase space of the theory, each point of which is a smooth function $u = u(x)$; the Hamiltonians of the hierarchy are the functionals $h_s^{\mathrm{KdV}}(u) := (4^s/(2s+1))\,\mathrm{Tr}(L^{\mathrm{KdV}})^{s+\frac{1}{2}}(u)$, and the companions of $L^{\mathrm{KdV}}(u) = \partial_{xx} + u$ are the operators $B_s^{\mathrm{KdV}}(u) := 4^{s-1}(L^{\mathrm{KdV}})_+^{s-\frac{1}{2}}(u)$ $(s = 0, 1, 2, \ldots)$, etc. Of course, when speaking of any KdV-type theory we denote with $+$ the projection on non-negative powers of $\partial_x$, and with Tr the well known Adler trace for pseudo-differential operators.

Now, following [5], we introduce the transformation

$$\Theta^\epsilon : \mathcal{U} \to \mathcal{A} \qquad u \mapsto \alpha = 1 + \epsilon^2 u \tag{2.1}$$

and employ it to pull back to the phase space $\mathcal{U}$ the geometrical structures and the Lax scheme of the $K_1$ system living on $\mathcal{A}$. Let us consider, in particular, the Lax operator

$$L^\epsilon(u) := L^\epsilon(\alpha)|_{\alpha=1+\epsilon^2 u} \tag{2.2}$$

and the pull back $Q^\epsilon_u$ of the Poisson tensor. Then, in the $\epsilon \mapsto 0$ limit, we obtain

$$L^\epsilon(u) = 2 + \epsilon^2(\partial_{xx} + u) + \mathrm{O}(\epsilon^3) = 2 + \epsilon^2 L^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^3) \tag{2.3}$$

$$Q^\epsilon_u = \frac{1}{\epsilon^3}\partial_x + \mathrm{O}\left(\frac{1}{\epsilon^2}\right) = \frac{1}{\epsilon^3} Q^{\mathrm{KdV}}(u) + \mathrm{O}\left(\frac{1}{\epsilon^2}\right). \tag{2.4}$$

The $K_1$ system also possesses a second Poisson tensor, and a recombination of the latter with $Q^\epsilon$ produces in the $\epsilon \mapsto 0$ limit the second KdV Poisson tensor; we give no more details about this fact, falling outside the purposes of the present paper.

All the Hamiltonians, the vector fields and the companion operators of KdV theory can be obtained as continuous limits of homologous $K_1$ objects. The main point in this construction is that the $K_1$ objects must be conveniently recombined before sending $\epsilon$ to zero; as observed in the introduction, the numerical coefficients of these recombinations could be determined recursively with the methods described in [6], but a different approach will be preferred here, which is more suitable for the extension to the $K_N$ systems with $N > 1$. Let us start with the following.

*Definition 2.1.* Let $z$ be an indeterminate, and

$$\lambda(z) := z + \frac{1}{z}. \tag{2.5}$$

For $s = 1, 2, 3, \ldots$ we will put

$$q_s(z) := \tfrac{1}{2}(\lambda^{2s}(z))_+ \tag{2.6}$$

the subscript $+$ denoting the projection on the non-negative powers of $z$. Also, we will put

$$p_0(\epsilon) := \frac{1}{2}\epsilon^2 \qquad p_s(\epsilon) := \frac{1}{2s}\binom{2s}{s}(1 + s\epsilon^2) \qquad (s = 1, 2, 3, \ldots). \tag{2.7}$$

In order to make the motivation for the previous definition clear, let us consider the $K_1$ Lax operator and its companions at the point $u = 0$ of the space $\mathcal{U}$ (corresponding to $\alpha = 1$ in equation (2.1)); it is easily found that

$$(L^\epsilon)^s(0) = \lambda^s(\Delta_\epsilon) \qquad A^\epsilon_s(0) = q_s(\Delta_\epsilon). \tag{2.8}$$

Furthermore, let us evaluate the $K_1$ Hamiltonians at $u = \mathrm{constant} = 1$ (i.e. $\alpha = 1 + \epsilon^2$); the traces appearing in their definition are easily computed, and it is found that

$$f^\epsilon_0(1) = \tfrac{1}{2}\log(1 + \epsilon^2) = p_0(\epsilon) + \mathrm{O}(\epsilon^3) \tag{2.9}$$

$$f^\epsilon_s(1) = \frac{1}{2s}\binom{2s}{s}(1 + \epsilon^2)^s = p_s(\epsilon) + \mathrm{O}(\epsilon^3) \qquad (s = 1, 2, 3\ldots) \tag{2.10}$$

so the polynomials $p_0(\epsilon), p_1(\epsilon), \ldots$ turn out to be second-order expansions of the Hamiltonians at $u = 1$.

*Definition 2.2.* Let $s \in \{0, 1, 2, \ldots\}$ be a fixed integer; consider two systems of coefficients $(c_{sj})_{j=-1,\ldots,s}$ and $(d_{sj})_{j=0,\ldots,s-1}$ (intending the second one to be empty if $s = 0$). These coefficients are said to satisfy the $C^{(1)}_s$ conditions if the following holds:

$s = 0$:

$$c_{0,-1} + c_{00}\, p_0(\epsilon) = \tfrac{1}{2}\epsilon^2 \tag{2.11}$$

$s \geqslant 1$:

$$\sum_{j=1}^{s} c_{sj} q_j(e^\epsilon) + \sum_{j=0}^{s-1} d_{sj}\lambda^j(e^\epsilon) = 4^{s-1}\epsilon^{2s-1} + \mathrm{O}(\epsilon^{2s}) \tag{2.12}$$

and

$$c_{s,-1} + \sum_{j=0}^{s} c_{sj} p_j(\epsilon) = 0. \tag{2.13}$$

The superscript (1) in $C_s^{(1)}$ marks the relations between the above prescriptions and the $K_1$ system, as already explained in the introduction. (For avoiding too many indexes, this superscript has been omitted in the coefficients $c_{sj}$, $d_{sj}$.)

For $s = 1, 2, 3, \ldots$, the $C_s^{(1)}$ conditions (2.12) and (2.13) give rise to a system of $2s+2$ linear, algebraic equations in the $2s+2$ unknowns $c_{sj}$ and $d_{sj}$; the equations are obtained by equating the coefficients of 1, $\epsilon, \epsilon^2, \ldots, \epsilon^{2s-1}$ in the two sides of equation (2.12), and setting to zero the coefficients of 1 and $\epsilon^2$ in the left-hand side of equation (2.13).

For a given integer $s$, let us consider a system of coefficients satisfying the $C_s^{(1)}$ conditions, and employ them to construct the linear combinations of operators, vector fields and Hamiltonians according to the following prescriptions:

$$B_s^\epsilon := \sum_{j=1}^{s} c_{sj} A_j^\epsilon + \sum_{j=0}^{s-1} d_{sj} (L^\epsilon)^j \tag{2.14}$$

$$Z_s^\epsilon := \sum_{j=1}^{s} c_{sj} X_j^\epsilon \tag{2.15}$$

$$h_s^\epsilon := c_{s,-1} + \sum_{j=0}^{s} c_{sj} f_j^\epsilon \tag{2.16}$$

(intending $B_0^\epsilon := 0$, $Z_0^\epsilon := 0$). We can regard these objects to be defined on either the $\mathcal{A}$ or the $\mathcal{U}$ phase space, thanks to the one-to-one correspondence (2.1). Let us work, in particular, on $\mathcal{U}$, and perform expansions in powers of $\epsilon$; where a shift operator $\Delta_{k\epsilon}$ occurs, we replace it with the expansion of $e^{k\epsilon\partial}$. With these prescriptions, the recombinations (2.14)–(2.16) give rise to power series in $\epsilon$; the coefficients are differential operators in the case of $A_s^\epsilon$ and differential polynomials (in $u$) in the case of $Z_s^\epsilon$. In the case of $h_s^\epsilon$, each coefficient of the $\epsilon$-expansion also involves a differential polynomial, integrated over $x$.

Let us return to the $C_s^{(1)}$ conditions, and interpret them from the viewpoint of these expansions. For $s \geqslant 1$, comparing equation (2.8) with (2.12) (and replacing formally $\epsilon$ with $\epsilon\partial$), we see that (2.12) means $B_s^\epsilon(0) = 4^{s-1}\epsilon^{2s-1}\partial^{2s-1} + \mathrm{O}(\epsilon^{2s})$; on the other hand, $4^{s-1}\partial^{2s-1}$ is just the $s$th KdV companion operator at $u = 0$, so we can write

$$B_s^\epsilon(0) = \epsilon^{2s-1} B_s^{\mathrm{KdV}}(0) + \mathrm{O}(\epsilon^{2s}). \tag{2.17}$$

As for the second $C_s^{(1)}$ condition (2.13), by comparison with (2.9), (2.10) and (2.16) we see that it amounts to the prescription

$$h_s^\epsilon(1) = \mathrm{O}(\epsilon^3) \tag{2.18}$$

for the recombined Hamiltonian.

Finally, equation (2.11) means

$$h_0^\epsilon(1) = \tfrac{1}{2}\epsilon^2 = \epsilon^2 h_0^{\mathrm{KdV}}(1). \tag{2.19}$$

Even though equations (2.17)–(2.19) simply involve the behaviour of the recombined $K_1$ operators and Hamiltonians at a particular point of the phase space $\mathcal{U}$, they allow us to infer the following, much stronger statement.

*Proposition 2.3.* Let $s \in \{0, 1, 2, \ldots\}$ be a fixed integer, and consider the recombinations (2.14)–(2.16), with coefficients satisfying the $C_s^{(1)}$ conditions. Then *at each point* $u \in \mathcal{U}$ it is

$$B_s^\epsilon(u) = \epsilon^{2s-1} B_s^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^{2s}) \tag{2.20}$$
$$Z_s^\epsilon(u) = \epsilon^{2s-1} Z_s^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^{2s}) \tag{2.21}$$
$$h_s^\epsilon(u) = \epsilon^{2s+2} h_s^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^{2s+3}). \tag{2.22}$$

The proof of these statements is omitted for brevity; it is similar to the (even more technical) proof of proposition 3.3, given explicitly in section 4 for the $K_2$ system.

Let us illustrate the recombination schemes for $s = 0, 1, 2, 3$, depending on

$$q_1(z) = 1 + \tfrac{1}{2}z^2 \qquad q_2(z) = 3 + 2z^2 + \tfrac{1}{2}z^4 \qquad q_3(z) = 10 + \tfrac{15}{2}z^2 + 3z^4 + \tfrac{1}{2}z^6 \tag{2.23}$$
$$\lambda^2(z) = z^2 + 2 + \frac{1}{z^2} \qquad \lambda^3(z) = z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} \tag{2.24}$$

and

$$p_0(\epsilon) = \tfrac{1}{2}\epsilon^2 \qquad p_1(\epsilon) = 1 + \epsilon^2 \qquad p_2(\epsilon) = \tfrac{3}{2} + 3\epsilon^2 \qquad p_3(\epsilon) = \tfrac{10}{3} + 10\epsilon^2. \tag{2.25}$$

Equation (2.11) has the unique solution

$$c_{0,-1} = 0 \qquad c_{00} = 1. \tag{2.26}$$

For $s = 1, 2, 3$ equations (2.12) and (2.13) have unique solutions, given respectively by

$$c_{1,-1} = -1 \qquad c_{10} = -2 \qquad c_{11} = 1 \qquad d_{10} = -\tfrac{3}{2} \tag{2.27}$$
$$c_{2,-1} = -\tfrac{9}{2} \qquad c_{20} = 6 \qquad c_{21} = -6 \qquad c_{22} = 1 \qquad d_{20} = \tfrac{15}{2} \qquad d_{21} = -2 \tag{2.28}$$
$$c_{3,-1} = -\tfrac{55}{3} \qquad c_{30} = -20 \qquad c_{31} = 30 \qquad c_{32} = -10 \qquad c_{33} = 1$$
$$d_{30} = -45 \qquad d_{31} = 32 \qquad d_{32} = -\tfrac{15}{2}. \tag{2.29}$$

The recombinations (2.14)–(2.16) with these coefficients behave as foreseen by proposition 2.3; for example, let us consider the case $s = 3$ in which

$$B_3^\epsilon = 30A_1^\epsilon - 10A_2^\epsilon + A_3^\epsilon - 45 + 32L^\epsilon - \tfrac{15}{2}(L^\epsilon)^2 \tag{2.30}$$
$$Z_3^\epsilon = 30X_1^\epsilon - 10X_2^\epsilon + X_3^\epsilon \tag{2.31}$$
$$h_3^\epsilon = -\tfrac{55}{3} - 20f_0^\epsilon + 30f_1^\epsilon - 10f_2^\epsilon + f_3^\epsilon. \tag{2.32}$$

Starting from table 1, one can apply the transformation (2.1) to the above recombinations and expand in $\epsilon$; to the lowest orders, it is found

$$\begin{aligned} B_3^\epsilon(u) &= 16\epsilon^5[\partial_{xxxxx} + \tfrac{5}{2}u\partial_{xxx} + \tfrac{15}{4}u_x\partial_{xx} + \tfrac{5}{8}(5u_{xx} + 3u^2)\partial_x + \tfrac{15}{16}(u_{xxx} + 2uu_x)] + \mathrm{O}(\epsilon^6) \\ &= \epsilon^5 B_3^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^6) \end{aligned} \tag{2.33}$$
$$Z_3^\epsilon(u) = \epsilon^5(u_{xxxxx} + 10uu_{xxx} + 20u_xu_{xx} + 30u^2u_x) + \mathrm{O}(\epsilon^6) = \epsilon^5 Z_3^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^6) \tag{2.34}$$
$$h_3^\epsilon(u) = \frac{1}{2}\epsilon^8 \int \mathrm{d}x\, (5u^4 - 10uu_x^2 + u_{xx}^2) + \mathrm{O}(\epsilon^9) = \epsilon^8 h_3^{\mathrm{KdV}}(u) + \mathrm{O}(\epsilon^9). \tag{2.35}$$

**Table 2.** $K_2$ system.

Lax formulation for the $s$th vector field $X^\epsilon_s$: $\quad \mathrm{d}L^\epsilon/\mathrm{d}t_s = [A^\epsilon_s L^\epsilon]$

$L^\epsilon(\boldsymbol{\alpha}) := \Delta_\epsilon + \alpha\Delta_{-\epsilon} + \rho\Delta_{-3\epsilon} \qquad A^\epsilon_s(\boldsymbol{\alpha}) := \frac{1}{3}(L^\epsilon)^{2s}_+(\boldsymbol{\alpha}) \qquad (s = 0, 1, 2, \ldots).$

Poisson tensor at a point $\boldsymbol{\alpha}$: $\quad Q^\epsilon_{\boldsymbol{\alpha}} : T^*_{\boldsymbol{\alpha}}\mathcal{A} \to T_{\boldsymbol{\alpha}}\mathcal{A}$

$$\begin{pmatrix} \dot\alpha \\ \dot\rho \end{pmatrix} = \frac{1}{3}\begin{pmatrix} Q^\epsilon_{\alpha\alpha} & Q^\epsilon_{\alpha\rho} \\ Q^\epsilon_{\rho\alpha} & Q^\epsilon_{\rho\rho} \end{pmatrix}\begin{pmatrix} \delta\alpha \\ \delta\rho \end{pmatrix}$$

$Q^\epsilon_{\alpha\alpha} := \rho_{[2\epsilon]}\Delta_{2\epsilon} + \alpha\alpha_{[\epsilon]}\Delta_\epsilon - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon} - \rho\Delta_{-2\epsilon}$

$Q^\epsilon_{\alpha\rho} := \alpha\rho_{[3\epsilon]}\Delta_{3\epsilon} + \alpha\rho_{[2\epsilon]}\Delta_{2\epsilon} - \alpha\rho - \alpha\rho_{[-\epsilon]}\Delta_{-\epsilon}$

$Q^\epsilon_{\rho\alpha} := \alpha_{[\epsilon]}\rho\Delta_\epsilon + \alpha\rho - \alpha_{[-2\epsilon]}\rho\Delta_{-2\epsilon} - \alpha_{[-3\epsilon]}\rho\Delta_{-3\epsilon}$

$Q^\epsilon_{\rho\rho} := \rho\rho_{[3\epsilon]}\Delta_{3\epsilon} + \rho\rho_{[2\epsilon]}\Delta_{2\epsilon} + \rho\rho_{[\epsilon]}\Delta_\epsilon - \rho\rho_{[-\epsilon]}\Delta_{-\epsilon} - \rho\rho_{[-2\epsilon]}\Delta_{-2\epsilon} - \rho\rho_{[-3\epsilon]}\Delta_{-3\epsilon}.$

Hamiltonian formulation: $\quad X^\epsilon_s = Q^\epsilon\,\mathrm{d}f^\epsilon_s \qquad (s = 0, 1, 2, \ldots).$

Hamiltonian functions:

$$f^\epsilon_0(\boldsymbol{\alpha}) := \frac{1}{2}\int \mathrm{d}x\,\log(-3\rho) \qquad f^\epsilon_s(\boldsymbol{\alpha}) := \frac{1}{2s}\,\mathrm{tr}(L^\epsilon)^{2s}(\boldsymbol{\alpha}) \qquad (s = 1, 2, 3, \ldots).$$

First companions of the Lax operator:

$A^\epsilon_0(\boldsymbol{\alpha}) = \frac{1}{3}$

$A^\epsilon_1(\boldsymbol{\alpha}) = \frac{1}{3}\Delta_{2\epsilon} + \frac{1}{3}(\alpha + \alpha_{[\epsilon]})$

$$\begin{aligned} A^\epsilon_2(\boldsymbol{\alpha}) = {} & \tfrac{1}{3}\Delta_{4\epsilon} + \tfrac{1}{3}(\alpha + \alpha_{[2\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{2\epsilon} \\ & + \tfrac{1}{3}(\alpha^2 + \alpha\alpha_{[-\epsilon]} + 2\alpha\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha^2_{[\epsilon]} + \rho + \rho_{[2\epsilon]} + \rho_{[3\epsilon]} + \rho_{[\epsilon]}) \end{aligned}$$

$$\begin{aligned} A^\epsilon_3(\boldsymbol{\alpha}) = {} & \tfrac{1}{3}\Delta_{6\epsilon} + \tfrac{1}{3}(\alpha + \alpha_{[5\epsilon]} + \alpha_{[2\epsilon]} + \alpha_{[4\epsilon]} + \alpha_{[3\epsilon]} + \alpha_{[\epsilon]})\Delta_{4\epsilon} \\ & + \tfrac{1}{3}(\alpha^2 + \alpha\alpha_{[2\epsilon]} + \alpha^2_{[2\epsilon]} + \alpha\alpha_{[-\epsilon]} + \alpha\alpha_{[3\epsilon]} + 2\alpha_{[2\epsilon]}\alpha_{[3\epsilon]} + \alpha_{[4\epsilon]}\alpha_{[3\epsilon]} + \alpha^2_{[3\epsilon]} + 2\alpha\alpha_{[\epsilon]} \\ & + 2\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha_{[3\epsilon]}\alpha_{[\epsilon]} + \alpha^2_{[\epsilon]} + \rho + \rho_{[5\epsilon]} + \rho_{[2\epsilon]} + \rho_{[4\epsilon]} + \rho_{[3\epsilon]} + \rho_{[\epsilon]})\Delta_{2\epsilon} \\ & + \tfrac{1}{3}(\alpha^3 + 2\alpha^2\alpha_{[-\epsilon]} + \alpha\alpha^2_{[-\epsilon]} + 3\alpha^2\alpha_{[\epsilon]} + 2\alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha^2_{[2\epsilon]}\alpha_{[\epsilon]} \\ & + 2\alpha\alpha_{[-\epsilon]}\alpha_{[\epsilon]} + \alpha_{[2\epsilon]}\alpha_{[3\epsilon]}\alpha_{[\epsilon]} + 3\alpha\alpha^2_{[\epsilon]} + 2\alpha_{[2\epsilon]}\alpha^2_{[\epsilon]} + \alpha^3_{[\epsilon]} + 2\alpha\rho + \alpha_{[-\epsilon]}\rho + 2\alpha_{[\epsilon]}\rho \\ & + 2\alpha\rho_{[2\epsilon]} + \alpha_{[2\epsilon]}\rho_{[2\epsilon]} + \alpha_{[-\epsilon]}\rho_{[2\epsilon]} + \alpha_{[3\epsilon]}\rho_{[2\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[2\epsilon]} + \alpha\rho_{[-\epsilon]} + \alpha_{[\epsilon]}\rho_{[4\epsilon]} \\ & + 2\alpha\rho_{[3\epsilon]} + \alpha_{[2\epsilon]}\rho_{[3\epsilon]} + \alpha_{[4\epsilon]}\rho_{[3\epsilon]} + \alpha_{[3\epsilon]}\rho_{[3\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[3\epsilon]} + 2\alpha\rho_{[\epsilon]} + \alpha_{[2\epsilon]}\rho_{[\epsilon]} \\ & + \alpha_{[-\epsilon]}\rho_{[\epsilon]} + 2\alpha_{[\epsilon]}\rho_{[\epsilon]} + \rho\alpha_{[-3\epsilon]} + \alpha\alpha_{[-\epsilon]}\alpha_{[-2\epsilon]} + \rho\alpha_{[-2\epsilon]} + \rho_{[\epsilon]}\alpha_{[-2\epsilon]}). \end{aligned}$$

First Hamiltonians after $f^\epsilon_0$:

$$f^\epsilon_1(\boldsymbol{\alpha}) = \int \mathrm{d}x\,\alpha$$

$$f^\epsilon_2(\boldsymbol{\alpha}) = \frac{1}{2}\int \mathrm{d}x(\alpha^2 + 2\alpha\alpha_{[\epsilon]} + 2\rho)$$

$$f^\epsilon_3(\boldsymbol{\alpha}) = \frac{1}{3}\int \mathrm{d}x(\alpha^3 + 3\alpha^2\alpha_{[\epsilon]} + 3\alpha\alpha^2_{[\epsilon]} + 3\alpha\alpha_{[\epsilon]}\alpha_{[2\epsilon]} + 3\alpha\rho + 3\alpha_{[\epsilon]}\rho + 3\alpha\rho_{[\epsilon]} + 3\alpha\rho_{[2\epsilon]} + 3\alpha\rho_{[3\epsilon]}).$$

First vector fields:

$$X^\epsilon_s(\boldsymbol{\alpha}) = \frac{1}{3}\begin{pmatrix} X^\epsilon_{s,\alpha}(\boldsymbol{\alpha}) \\ X^\epsilon_{s,\rho}(\boldsymbol{\alpha}) \end{pmatrix}$$

where, for $s = 0, 1, 2, 3$:

$X^\epsilon_{0,\alpha}(\boldsymbol{\alpha}) := 0 \qquad X^\epsilon_{0,\rho}(\boldsymbol{\alpha}) := 0$

$X^\epsilon_{1,\alpha}(\boldsymbol{\alpha}) := -\alpha\alpha_{[-\epsilon]} + \alpha\alpha_{[\epsilon]} - \rho + \rho_{[2\epsilon]}$

$X^\epsilon_{1,\rho}(\boldsymbol{\alpha}) := \alpha\rho - \alpha_{[-2\epsilon]}\rho - \alpha_{[-3\epsilon]}\rho + \alpha_{[\epsilon]}\rho$

$$\begin{aligned} X^\epsilon_{2,\alpha}(\boldsymbol{\alpha}) := {} & -\alpha^2\alpha_{[-\epsilon]} - \alpha\alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} - \alpha\alpha^2_{[-\epsilon]} + \alpha^2\alpha_{[\epsilon]} + \alpha\alpha_{[2\epsilon]}\alpha_{[\epsilon]} + \alpha\alpha^2_{[\epsilon]} - \alpha\rho - \alpha_{[-2\epsilon]}\rho - \alpha_{[-3\epsilon]}\rho - \alpha_{[-\epsilon]}\rho \\ & + \alpha\rho_{[2\epsilon]} + \alpha_{[2\epsilon]}\rho_{[2\epsilon]} + \alpha_{[3\epsilon]}\rho_{[2\epsilon]} + \alpha_{[\epsilon]}\rho_{[2\epsilon]} - \alpha\rho_{[-\epsilon]} + \alpha\rho_{[3\epsilon]} \end{aligned}$$

$$\begin{aligned} X^\epsilon_{2,\rho}(\boldsymbol{\alpha}) := {} & \alpha^2\rho - \alpha^2_{[-2\epsilon]}\rho - 2\alpha_{[-2\epsilon]}\alpha_{[-3\epsilon]}\rho - \alpha_{[-4\epsilon]}\alpha_{[-3\epsilon]}\rho - \alpha^2_{[-3\epsilon]}\rho + \alpha\alpha_{[-\epsilon]}\rho - \alpha_{[-2\epsilon]}\alpha_{[-\epsilon]}\rho + 2\alpha\alpha_{[\epsilon]}\rho \\ & + \alpha_{[2\epsilon]}\alpha_{[\epsilon]}\rho + \alpha^2_{[\epsilon]}\rho + \rho\rho_{[2\epsilon]} - \rho\rho_{[-2\epsilon]} - \rho\rho_{[-3\epsilon]} - \rho\rho_{[-\epsilon]} + \rho\rho_{[3\epsilon]} + \rho\rho_{[\epsilon]} \end{aligned}$$

**Table 2.** (Continued)

$X^{\epsilon}_{3,\alpha}(\boldsymbol{\alpha}) := Q^{\epsilon}_{\alpha\alpha}(\delta\alpha) + Q^{\epsilon}_{\alpha\rho}(\delta\rho)$

$X^{\epsilon}_{3,\rho}(\boldsymbol{\alpha}) := Q^{\epsilon}_{\rho\alpha}(\delta\alpha) + Q^{\epsilon}_{\rho\rho}(\delta\rho)$

with

$\delta\alpha := \alpha^2 + 2\alpha\alpha_{[\epsilon]} + \alpha^2_{[-\epsilon]} + \alpha^2_{[\epsilon]} + 2\alpha_{[-\epsilon]}\alpha + \alpha_{[\epsilon]}\alpha_{[2\epsilon]} + \alpha_{[-\epsilon]}\alpha_{[\epsilon]} + \alpha_{[-2\epsilon]}\alpha_{[-\epsilon]} + \rho + \rho_{[-\epsilon]} + \rho_{[\epsilon]} + \rho_{[2\epsilon]} + \rho_{[3\epsilon]}$

$\delta\rho := \alpha + \alpha_{[\epsilon]} + \alpha_{[-\epsilon]} + \alpha_{[-2\epsilon]} + \alpha_{[-3\epsilon]}$

(these are the components of $\mathrm{d}f^{\epsilon}_3$).

## 3. The $K_2$ system and Boussinesq theory

The main facts about the $K_2$ system are described in table 2. The interpolated version of this system is a theory in two field variables $\alpha_1 := \alpha$ and $\alpha_2 := \rho$. The phase space $\mathcal{A}$ is a set of such pairs $\boldsymbol{\alpha} = (\alpha, \rho)$; at each point the tangent and cotangent spaces $T_{\boldsymbol{\alpha}}\mathcal{A}$ and $T^*_{\boldsymbol{\alpha}}\mathcal{A}$ are represented as sets of pairs $\dot{\boldsymbol{\alpha}} = (\dot{\alpha}, \dot{\rho})$ and $\delta\boldsymbol{\alpha} = (\delta\alpha, \delta\rho)$, with the duality form

$$\langle \delta\boldsymbol{\alpha}, \dot{\boldsymbol{\alpha}}\rangle := \int \mathrm{d}x\, (\delta\alpha\dot{\alpha} + \delta\rho\dot{\rho}). \tag{3.1}$$

Table 2 reports the Lax formulation, the Poisson structure and the explicit expressions of the first elements in the associated hierarchy.

Concerning the Boussinesq theory in the field variables $u_1 := u$, $u_2 := v$, we collect hereafter the basic elements, mainly in order to fix some notational standards. We denote with $\mathcal{U}$ the phase space, whose points are pairs $\boldsymbol{u} = (u, v)$; with our notation, the familiar Boussinesq equation $u_{\tau\tau} + \frac{1}{3}u_{xxxx} + \frac{2}{3}(u^2)_{xx} = 0$ arises from the evolution equation corresponding to the vector field $Z^{\mathrm{Bou}}_3$.

In order to connect the theories described in tables 2 and 3, let us consider the transformation $\Theta^{\epsilon} : \mathcal{U} \to \boldsymbol{a}$, $\boldsymbol{u} = (u, v) \mapsto \boldsymbol{\alpha} = (\alpha, \rho)$, where

$$\begin{aligned} \alpha &= 2 + \tfrac{2}{3}\epsilon^2 u \\ \rho &= -\tfrac{1}{3} - \tfrac{2}{3}\epsilon^2 u + \tfrac{4}{3}\epsilon^3 v. \end{aligned} \tag{3.2}$$

This is the specialization to the case $N = 2$ of the general transformation introduced in [5], in order to obtain the $sl(N + 1)$ Lax operator from the limit of the $K_N$ Lax operator. In fact, setting

$$L^{\epsilon}(\boldsymbol{u}) := L^{\epsilon}(\boldsymbol{\alpha})|_{\boldsymbol{\alpha}=\Theta^{\epsilon}(\boldsymbol{u})} \tag{3.3}$$

one easily finds, in the $\epsilon \mapsto 0$ limit,

$$L^{\epsilon}(\boldsymbol{u}) = \tfrac{8}{3} + \tfrac{4}{3}\epsilon^3(\partial_{xxx} + u\partial_x + v) + \mathrm{O}(\epsilon^4) = \tfrac{8}{3} + \tfrac{4}{3}\epsilon^3 L^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^4). \tag{3.4}$$

Now, let us go on in the analysis of the $\epsilon \mapsto 0$ limit, and employ the transformation $\Theta^{\epsilon}$ to pull back onto $\mathcal{U}$ the Poisson tensor of table 2; in this way we obtain, at each point $\boldsymbol{u}$, the Poisson tensor

$$Q^{\epsilon}_{\boldsymbol{u}} := (T_{\boldsymbol{u}}\Theta^{\epsilon})^{-1} Q^{\epsilon}_{\boldsymbol{\alpha}}|_{\boldsymbol{\alpha}=\Theta^{\epsilon}(\boldsymbol{u})}(T^*_{\boldsymbol{u}}\Theta^{\epsilon})^{-1}. \tag{3.5}$$

The above formula contains the inverses of the tangent and cotangent maps $T_{\boldsymbol{u}}\Theta^{\epsilon} : T_{\boldsymbol{u}}\mathcal{U} \to T_{\boldsymbol{\alpha}}\mathcal{A}$ and $T^*_{\boldsymbol{u}}\Theta^{\epsilon} : T^*_{\boldsymbol{\alpha}}\mathcal{A} \to T^*_{\boldsymbol{u}}\mathcal{U}$; explicit computations give

$$Q^{\epsilon}_{\boldsymbol{u}} = \begin{pmatrix} 3/2\epsilon^2 & 0 \\ 3/4\epsilon^3 & 3/4\epsilon^3 \end{pmatrix} \begin{pmatrix} \frac{1}{3}Q^{\epsilon}_{\alpha\alpha} & \frac{1}{3}Q^{\epsilon}_{\alpha\rho} \\ \frac{1}{3}Q^{\epsilon}_{\rho\alpha} & \frac{1}{3}Q^{\epsilon}_{\rho\rho} \end{pmatrix} \begin{pmatrix} 3/2\epsilon^2 & 3/4\epsilon^3 \\ 0 & 3/4\epsilon^3 \end{pmatrix} \tag{3.6}$$

**Table 3.** Boussinesq theory.

Lax formulation for the $s$th vector field $Z_s^{\mathrm{Bou}}$: $\quad \mathrm{d}L^{\mathrm{Bou}}/\mathrm{d}\tau_s = [B_s^{\mathrm{Bou}}, L^{\mathrm{Bou}}]$

$L^{\mathrm{Bou}}(\boldsymbol{u}) := \partial_{xxx} + u\partial_x + v \qquad B_s^{\mathrm{Bou}} := 3^{[s/2]-1}(L^{\mathrm{Bou}})_+^{\frac{1}{3}[3s/2]-\frac{2}{3}} \qquad (s = 0, 1, 2, \ldots).$

(of course, [.] stands for the integer part).

Poisson tensor at a point $\boldsymbol{u}$: $\quad Q_{\boldsymbol{u}}^{\mathrm{Bou}} : T_{\boldsymbol{u}}^*\mathcal{U} \to T_{\boldsymbol{u}}\mathcal{U}$

$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix}\begin{pmatrix} \delta u \\ \delta v \end{pmatrix}.$

Hamiltonian formulation: $\quad Z_s^{\mathrm{Bou}} = Q^{\mathrm{Bou}}\,\mathrm{d}h_s^{\mathrm{Bou}} \qquad (s = 0, 1, 2, \ldots).$

Hamiltonian functions:

$h_s^{\mathrm{Bou}}(\boldsymbol{u}) := \frac{3^{[s/2]+1}}{[3s/2]+1}\,\mathrm{Tr}(L^{\mathrm{Bou}})(\boldsymbol{u})^{\frac{1}{3}[3s/2]+\frac{1}{3}}.$

First non-zero companions of the Lax operator:

$B_2^{\mathrm{Bou}}(\boldsymbol{u}) = \partial_x$

$B_3^{\mathrm{Bou}}(\boldsymbol{u}) := \partial_{xx} + \frac{2}{3}u.$

First Hamiltonians:

$h_0^{\mathrm{Bou}}(\boldsymbol{u}) = \int \mathrm{d}x\, u$

$h_1^{\mathrm{Bou}}(\boldsymbol{u}) = \int \mathrm{d}x\, v$

$h_2^{\mathrm{Bou}}(\boldsymbol{u}) = \int \mathrm{d}x\, uv$

$h_3^{\mathrm{Bou}}(\boldsymbol{u}) = \frac{1}{9}\int \mathrm{d}x(-u^3 + 9v^2 - 9u_x v + 3u_x^2)$

$h_4^{\mathrm{Bou}}(\boldsymbol{u}) = \frac{1}{9}\int \mathrm{d}x(-u^4 + 9u{u_x}^2 - 3{u_{xx}}^2 - 18uu_x v - 9u_{xxx}v + 18uv^2 - 9{v_x}^2).$

First non-zero vector fields:

$Z_2^{\mathrm{Bou}}(\boldsymbol{u}) = \begin{pmatrix} u_x \\ v_x \end{pmatrix}$

$Z_3^{\mathrm{Bou}}(\boldsymbol{u}) = \begin{pmatrix} -u_{xx} + 2v_x \\ -\frac{2}{3}u_{xxx} - \frac{2}{3}uu_x + v_{xx} \end{pmatrix}.$

where $Q^{\epsilon}_{\alpha\alpha}$, etc are the matrix elements reported in table 2, with $\alpha$ and $\rho$ given by equation (3.2). The $\epsilon$-expansion of this tensor gives

$$Q^{\epsilon}_{\boldsymbol{u}} = \frac{1}{3\epsilon^4}\begin{pmatrix} 0 & \partial_x \\ \partial_x & 0 \end{pmatrix} + \mathrm{O}\left(\frac{1}{\epsilon^3}\right) = \frac{1}{\epsilon^4}Q^{\mathrm{Bou}}_{\boldsymbol{u}} + \mathrm{O}\left(\frac{1}{\epsilon^3}\right). \tag{3.7}$$

In order to obtain (for $\epsilon \mapsto 0$) the Hamiltonians, the vector fields and the companion operators of the Boussinesq theory, we recombine the homologous objects of the $K_2$ system with the method already described in the previous section for the case $N = 1$.

*Definition 3.1.* Let $z$ be an indeterminate, and

$$\lambda(z) := z + \frac{2}{z} - \frac{1}{3z^3} \qquad \mu(z) := \frac{2}{3z} - \frac{2}{3z^3}. \tag{3.8}$$

For $s = 1, 2, 3, \ldots,$ we will put

$$q_s(z) := \tfrac{1}{3}(\lambda^{2s}(z))_+ \tag{3.9}$$

and

$$p_s(\epsilon) := \frac{1}{2s}\,\mathrm{rem}(\lambda^{2s}(z)) + \mathrm{rem}(\mu(z)\lambda^{2s-1}(z))\epsilon^2 + \mathrm{rem}\left(\frac{4\lambda^{2s-1}(z)}{3z^3}\right)\epsilon^3 \tag{3.10}$$

where the 'reminder' rem is the coefficient of the zero-order term in $z$. Also, we will put

$$p_0(\epsilon) := \epsilon^2 - 2\epsilon^3. \tag{3.11}$$

Let us consider the Lax operator for the Kupershmidt system and its companions, evaluated at $u = v = 0$ (i.e. $\alpha = 2$, $\rho = -\frac{1}{3}$ according to equation (3.2)); with the notation just defined, we can write

$$(L^\epsilon)^s(0,0) = \lambda^s(\Delta_\epsilon) \qquad A_s^\epsilon(0,0) = q_s(\Delta_\epsilon). \tag{3.12}$$

The cubic polynomials in equation (3.10) and (3.11) are obtained by evaluating the $K_2$ Hamiltonians at $u = v = 1$ (i.e. $\alpha = 2 + \frac{2}{3}\epsilon^2$, $\rho = -\frac{1}{3} - \frac{2}{3}\epsilon^2 + \frac{4}{3}\epsilon^3$) and expanding up to order three in $\epsilon$:

$$f_s^\epsilon(1,1) = p_s(\epsilon) + \mathrm{O}(\epsilon^4) \qquad (s = 0, 1, 2, \ldots). \tag{3.13}$$

In order to prove this statement we observe that, writing $z$ in place of $\Delta_\epsilon$, for $s = 1, 2, 3, \ldots$ we have

$$f_s^\epsilon(1,1) = \frac{1}{2s}\,\mathrm{rem}\left(\lambda(z) + \mu(z)\epsilon^2 + \frac{4}{3z^3}\epsilon^3\right)^{2s}. \tag{3.14}$$

However,

$$\left(\lambda(z) + \mu(z)\epsilon^2 + \frac{4}{3z^3}\epsilon^3\right)^{2s} = \lambda^{2s}(z) + 2s\mu(z)\lambda^{2s-1}(z)\epsilon^2 + \frac{8s\lambda^{2s-1}(z)}{3z^3}\epsilon^3 + \mathrm{O}(\epsilon^4) \tag{3.15}$$

and so, taking the reminder, we obtain equation (3.13); the validity of this equation for $s = 0$ can also be checked easily.

The recombination rules in which we are interested are based on the following.

*Definition 3.2.* Let $s \in \{0, 1, 2, \ldots\}$ be a given integer, and consider two systems of coefficients $(c_{sj})_{j=-1,\ldots,s}$ and $(d_{sj})_{j=0,\ldots,[s/2]-1}$ (intending the second one to be empty if $s = 0$ or $s = 1$). These coefficients are said to satisfy the $C_s^{(2)}$ conditions if the following holds:

$s = 0$:

$$c_{0,-1} + c_{00}p_0(\epsilon) = \epsilon^2 + \mathrm{O}(\epsilon^3) \tag{3.16}$$

$s = 1$:

$$c_{1,-1} + c_{10}p_0(\epsilon) + c_{11}p_1(\epsilon) = \epsilon^3 \tag{3.17}$$

$s \geqslant 2$:

$$\sum_{j=1}^{s} c_{sj}q_j(e^\epsilon) + \sum_{j=0}^{[s/2]-1} d_{sj}\lambda^j(e^\epsilon) = 3^{[s/2]-1}\epsilon^{[3s/2]-2} + \mathrm{O}(\epsilon^{[3s/2]-1}) \tag{3.18}$$

and

$$c_{s,-1} + \sum_{j=0}^{s} c_{sj}p_j(\epsilon) = 0. \tag{3.19}$$

For each integer $s$, the $C_s^{(2)}$ conditions correspond to a system of as many equations as the unknown coefficients. This is clear for $s = 0$ and 1. For $s \geqslant 2$, we must equate the coefficients of all powers of $\epsilon$ from 0 to $[3s/2] - 2$ in equation (3.18), and set to zero the coefficients of 1, $\epsilon^2$, $\epsilon^3$ in equation (3.19); therefore, the total number of equations to be satisfied is $1 + ([3s/2] - 2) + 3 = [3s/2] + 2$, which is just the number of coefficients $(c_{sj})$ and $(d_{sj})$.

From a given system of coefficients satisfying the $C_s^{(2)}$ conditions for some $s$, let us define the following linear combinations of operators, vector fields and Hamiltonians:

$$B_s^\epsilon := \sum_{j=1}^{s} c_{sj} A_j^\epsilon + \sum_{j=0}^{[s/2]-1} d_{sj} (L^\epsilon)^j \tag{3.20}$$

$$Z_s^\epsilon := \sum_{j=1}^{s} c_{sj} X_j^\epsilon \tag{3.21}$$

$$h_s^\epsilon := c_{s,-1} + \sum_{j=0}^{s} c_{sj} f_j^\epsilon \tag{3.22}$$

(intending $B_0^\epsilon := 0$, $Z_0^\epsilon := 0$). Then, we have by construction

$$\frac{\mathrm{d}L^\epsilon}{\mathrm{d}\theta_s} = [B_s^\epsilon, L^\epsilon] \tag{3.23}$$

where $\mathrm{d}/\mathrm{d}\theta_s$ is the derivative along the vector field $Z_s^\epsilon$, and

$$Z_s^\epsilon = Q^\epsilon \, \mathrm{d}h_s^\epsilon. \tag{3.24}$$

If $s \geqslant 2$, let us evaluate the recombined operator at $\boldsymbol{u} = (0, 0)$ and expand it in powers of $\epsilon$. Recalling that $\Delta_{k\epsilon} = e^{k\epsilon\partial}$, and replacing $\epsilon$ with $\epsilon\partial$ in equation (3.18), we recognize this equation to be equivalent to

$$B_s^\epsilon(0, 0) = 3^{[s/2]-1} \epsilon^{[3s/2]-2} \partial^{[3s/2]-2} + \mathrm{O}(\epsilon^{[3s/2]-1}) = \epsilon^{[3s/2]-2} B_s^{\mathrm{Bou}}(0, 0) + \mathrm{O}(\epsilon^{[3s/2]-1}). \tag{3.25}$$

This clarifies the meaning of the first $C_s^{(2)}$ condition; the second one, corresponding to equation (3.19), means that the recombined Hamiltonian behaves as follows at $\boldsymbol{u} = (1, 1)$:

$$h_s^\epsilon(1, 1) = \mathrm{O}(\epsilon^4). \tag{3.26}$$

The $C_s^{(2)}$ conditions for $s = 0$ and 1 consist only of equations (3.16) and (3.17), whose meaning is

$$h_0^\epsilon(1, 1) = \epsilon^2 + \mathrm{O}(\epsilon^3) = \epsilon^2 h_0^{\mathrm{Bou}}(1, 1) + \mathrm{O}(\epsilon^3) \tag{3.27}$$

$$h_1^\epsilon(1, 1) = \epsilon^3 + \mathrm{O}(\epsilon^4) = \epsilon^3 h_1^{\mathrm{Bou}}(1, 1) + \mathrm{O}(\epsilon^4). \tag{3.28}$$

Just as in the $N = 1$ theory considered in the previous section, the above statements on the recombined operator and Hamiltonian at a particular point of the space $\mathcal{U}$ control the global behaviour of the recombinations (3.20)–(3.22):

*Proposition 3.3.* Let $s \in \{0, 1, 2, \ldots\}$ be a fixed integer. The $C_s^{(2)}$ conditions imply that *at each point* $\boldsymbol{u} \in \mathcal{U}$ it is

$$B_s^\epsilon(\boldsymbol{u}) = \epsilon^{[3s/2]-2} B_s^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^{[3s/2]-1}) \tag{3.29}$$

$$Z_s^\epsilon(\boldsymbol{u}) = \epsilon^{[3s/2]-2} Z_s^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^{[3s/2]-1}) \tag{3.30}$$

$$h_s^\epsilon(\boldsymbol{u}) = \epsilon^{[3s/2]+2} h_s^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^{[3s/2]+3}). \tag{3.31}$$

The next section will be fully devoted to the proof of this proposition. Independent of the general argument proposed therein, statements (3.29)–(3.31) can be checked by direct computation for the lowest values of $s$, say up to $s = 4$. To perform the calculations, it is necessary to know that

$$q_1(z) = \tfrac{1}{3}z^2 + \tfrac{4}{3} \qquad q_2(z) = \tfrac{1}{3}z^4 + \tfrac{8}{3}z^2 + \tfrac{68}{9}$$
$$q_3(z) = \tfrac{1}{3}z^6 + 4z^4 + \tfrac{58}{3}z^2 + \tfrac{140}{3}$$
$$q_4(z) = \tfrac{1}{3}z^8 + \tfrac{16}{3}z^6 + \tfrac{328}{9}z^4 + \tfrac{1232}{9}z^2 + \tfrac{8092}{27} \tag{3.32}$$
$$p_0(\epsilon) = \epsilon^2 - 2\epsilon^3 \qquad p_1(\epsilon) = 2 + \tfrac{2}{3}\epsilon^2 \qquad p_2(\epsilon) = \tfrac{17}{3} + \tfrac{10}{3}\epsilon^2 + \tfrac{4}{3}\epsilon^3$$
$$p_3(\epsilon) = \tfrac{70}{3} + \tfrac{170}{9}\epsilon^2 + \tfrac{40}{3}\epsilon^3 \qquad p_4(\epsilon) = \tfrac{2023}{18} + \tfrac{1022}{9}\epsilon^2 + \tfrac{980}{9}\epsilon^3. \tag{3.33}$$

The unique system of coefficients satisfying the $C_s^{(2)}$ condition for $s = 0$ (i.e. equation (3.16)) is

$$c_{0,-1} = 0 \qquad c_{00} = 1. \tag{3.34}$$

For $s = 1$ we must satisfy equation (3.17), yielding the unique solution

$$c_{1,-1} = -\tfrac{3}{2} \qquad c_{10} = -\tfrac{1}{2} \qquad c_{11} = \tfrac{3}{4}. \tag{3.35}$$

Let us consider the recombined Hamiltonians

$$h_0^\epsilon = f_0^\epsilon \tag{3.36}$$
$$h_1^\epsilon = -\tfrac{3}{2} - \tfrac{1}{2}f_0^\epsilon + \tfrac{3}{4}f_1^\epsilon. \tag{3.37}$$

Expressing the summands as in table 2, performing the substitution (3.2) and expanding in $\epsilon$, we find

$$h_0^\epsilon(\boldsymbol{u}) = \frac{1}{2}\int \mathrm{d}x\, \log(1 + 2\epsilon^2 u - 4\epsilon^3 v) = \epsilon^2 \int \mathrm{d}x\, u + \mathrm{O}(\epsilon^3) = \epsilon^2 h_0^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^3) \tag{3.38}$$

$$\begin{aligned} h_1^\epsilon(\boldsymbol{u}) &= -\frac{3}{2} - \frac{1}{4}\int \mathrm{d}x\, \log(1 + 2\epsilon^2 u - 4\epsilon^3 v) + \frac{3}{4}\int \mathrm{d}x\, \left(2 + \frac{2}{3}\epsilon^2 u\right) \\ &= \epsilon^3 \int \mathrm{d}x\, v + \mathrm{O}(\epsilon^4) = \epsilon^3 h_1^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^4) \end{aligned} \tag{3.39}$$

in agreement with proposition 3.3. The other statements in this proposition are trivially satisfied for $s = 0$ and 1 (note that $B_s^{\mathrm{Bou}}(\boldsymbol{u}) = 0$ and $Z_s^{\mathrm{Bou}}(\boldsymbol{u}) = 0$ in these cases).

For $s \geqslant 2$, the $C_s^{(2)}$ conditions are represented by equations (3.18) and (3.19); we report the solutions up to $s = 4$, which are unique. For $s = 2$, the solution is

$$c_{2,-1} = \tfrac{19}{8} \qquad c_{20} = \tfrac{1}{4} \qquad c_{21} = -\tfrac{9}{4} \qquad c_{22} = \tfrac{3}{8} \qquad d_{20} = -\tfrac{5}{24}. \tag{3.40}$$

For $s = 3$ we find

$$\begin{aligned} &c_{3,-1} = -\tfrac{37}{8} \qquad c_{30} = -\tfrac{1}{4} \qquad c_{31} = \tfrac{57}{8} \qquad c_{32} = -\tfrac{21}{8} \qquad c_{33} = \tfrac{9}{40} \\ &\qquad d_{30} = \tfrac{1}{120} \end{aligned} \tag{3.41}$$

and, finally, the solution for $s = 4$ is

$$\begin{aligned} &c_{4,-1} = \tfrac{147}{32} \qquad c_{40} = \tfrac{1}{8} \qquad c_{41} = -\tfrac{45}{4} \qquad c_{42} = \tfrac{57}{8} \qquad c_{43} = -\tfrac{27}{20} \qquad c_{44} = \tfrac{9}{112} \\ &\qquad d_{40} = \tfrac{67}{48} \qquad d_{41} = -\tfrac{18}{35}. \end{aligned} \tag{3.42}$$

These coefficients can be employed to construct the recombinations (3.20)–(3.22); for example, for $s = 3$ we have

$$B_3^\epsilon = \tfrac{57}{8}A_1^\epsilon - \tfrac{21}{8}A_2^\epsilon + \tfrac{9}{40}A_3^\epsilon + \tfrac{1}{120} \tag{3.43}$$

$$Z_3^\epsilon = \tfrac{57}{8}X_1^\epsilon - \tfrac{21}{8}X_2^\epsilon + \tfrac{9}{40}X_3^\epsilon \tag{3.44}$$

$$h_3^\epsilon = -\tfrac{37}{8} - \tfrac{1}{4}f_0^\epsilon + \tfrac{57}{8}f_1^\epsilon - \tfrac{21}{8}f_2^\epsilon + \tfrac{9}{40}f_3^\epsilon. \tag{3.45}$$

On application of the transformation (3.2), the lowest-order expansions in $\epsilon$ are

$$B_3^\epsilon(\boldsymbol{u}) = \epsilon^2 B_3^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^3) \tag{3.46}$$

$$Z_3^\epsilon(\boldsymbol{u}) = \epsilon^2 Z_3^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^3) \tag{3.47}$$

$$h_3^\epsilon(\boldsymbol{u}) = \epsilon^6 h_3^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^7) \tag{3.48}$$

(the Boussinesq objects being as in table 3).

Similar statements hold for $s = 4$; for example, the recombined Hamiltonian

$$h_4^\epsilon = \tfrac{147}{32} + \tfrac{1}{8}f_0^\epsilon - \tfrac{45}{4}f_1^\epsilon + \tfrac{57}{8}f_2^\epsilon - \tfrac{27}{20}f_3^\epsilon + \tfrac{9}{112}f_4^\epsilon \tag{3.49}$$

has the expansion

$$h_4^\epsilon(\boldsymbol{u}) = \epsilon^8 h_4^{\mathrm{Bou}}(\boldsymbol{u}) + \mathrm{O}(\epsilon^9). \tag{3.50}$$

The justification of equations (3.46)–(3.48) and (3.50) by direct computation (rather than by the theoretical arguments of section 4) is a task of a great computational complexity. Using an automatic manipulator is essential for this test; this is in fact what we actually did†.

## 4. Proof of proposition 3.3

For $s = 0$ and 1, the verification of equations (3.29)–(3.31) rests on the direct computation sketched in section 3. From now on, and up to the end of the present section, $s$ is an arbitrary integer greater than or equal to two. The proof will be divided into three steps; in spite of technical differences, some basic ideas already employed in the previous paper [1] can be recognized.

*Step 1: $\epsilon$-expansions of the $K_2$ structures and their recombinations.* It is evident that only non-negative powers of $\epsilon$ will appear in the developments of the companion operators $A_s^\epsilon$, the powers of $L^\epsilon$ and the recombinations (3.20). Let $q \geqslant 0$ be the exponent of the lowest-order term in the $\epsilon$-expansion of $B_s^\epsilon$; then we can write

$$B_s^\epsilon(\boldsymbol{u}) = \sum_{m=q}^{+\infty} \epsilon^m B_{s,m}(\boldsymbol{u}) \tag{4.1}$$

where $B_{s,m}(\boldsymbol{u})$ is, for each $m$, a differential operator with coefficients depending polynomially on the fields $\boldsymbol{u} = (u, v)$ and their $x$-derivatives. The non-negative integer $q = q(s)$ will be determined in step 2; note that, by definition, the operator $B_{s,q}(\boldsymbol{u})$ is non-zero, at least at some point $\boldsymbol{u} \in \mathcal{U}$. For the $K_2$ Lax operator, we have the expansion

$$L^\epsilon(\boldsymbol{u}) = \frac{8}{3} + \sum_{m=3}^{+\infty} \epsilon^m L_m(\boldsymbol{u}) \tag{4.2}$$

† The explicit expressions of the objects to be recombined for $s \leqslant 3$ are those written in table 2. For the sake of brevity, the printout for $f_4^\epsilon$ has not been reported in the table.

where each term $L_m(\boldsymbol{u})$ is also a differential operator, depending polynomially on $\boldsymbol{u}$ and its derivatives; in particular,

$$L_3(\boldsymbol{u}) = \tfrac{4}{3} L^{\mathrm{Bou}}(\boldsymbol{u}) \tag{4.3}$$

(recall equation (3.4)). Now let us consider the Hamiltonians: $f_0^\epsilon(\boldsymbol{\alpha})$ is (the integral over $x$ of) $\log(-3\rho)$ while $f_j^\epsilon(\boldsymbol{\alpha})$, for $j \geqslant 1$, is (the integral of) a polynomial expression in $\alpha$, $\rho$ and their shifts.

Expressing $\alpha$ and $\rho$ via equation (3.2), we see that their shifts of any order $k$ are

$$\begin{aligned} \alpha_{[k\epsilon]} &= 2 + \tfrac{2}{3}\epsilon^2 u_{[k\epsilon]} = 2 + \tfrac{2}{3}\epsilon^2 u + \tfrac{2}{3}\epsilon^3 k u_x + \mathrm{O}(\epsilon^4) \\ \rho_{[k\epsilon]} &= -\tfrac{1}{3} - \tfrac{2}{3}\epsilon^2 u_{[k\epsilon]} + \tfrac{4}{3}\epsilon^3 v_{[k\epsilon]} = -\tfrac{1}{3} - \tfrac{2}{3}\epsilon^2 u - \tfrac{2}{3}\epsilon^3 k u_x + \tfrac{4}{3}\epsilon^3 v + \mathrm{O}(\epsilon^4). \end{aligned} \tag{4.4}$$

This implies that the expansion up to order three of any of the Hamiltonians $f_j^\epsilon$ is of the form

$$\text{constant} + \text{constant } \epsilon^2 \int \mathrm{d}x\, u + \text{constant } \epsilon^3 \int \mathrm{d}x\, v + \text{constant } \epsilon^3 \int \mathrm{d}x\, u_x + \mathrm{O}(\epsilon^4).$$

The third integral vanishes due to the presence of a total $x$-derivative. Similar conclusions hold for the recombination $h_s^\epsilon$, so there are three real constants $\eta_s$, $\kappa_s$ and $\chi_s$ such that

$$h_s^\epsilon(\boldsymbol{u}) = \eta_s + \kappa_s \epsilon^2 \int \mathrm{d}x\, u + \chi_s \epsilon^3 \int \mathrm{d}x\, v + \mathrm{O}(\epsilon^4). \tag{4.5}$$

Comparing this result with equation (3.26) (expressing the second $C_s^{(2)}$ condition) we easily infer that $\eta_s = \kappa_s = \chi_s = 0$, i.e. $h_s^\epsilon(\boldsymbol{u}) = \mathrm{O}(\epsilon^4)$. More explicitly, we can write

$$h_s^\epsilon(\boldsymbol{u}) = \sum_{m=4}^{+\infty} \epsilon^m h_{s,m}(\boldsymbol{u}) \tag{4.6}$$

where each term $h_{s,m}(\boldsymbol{u})$ is (the integral over $x$ of) a polynomial density in the fields $\boldsymbol{u}$ and their $x$-derivatives.

We go on expanding and consider the $K_2$ Poisson tensor. Recalling equation (3.7), we can write

$$Q_{\boldsymbol{u}}^\epsilon = \sum_{m=-4}^{+\infty} \epsilon^m Q_{m,\boldsymbol{u}} \tag{4.7}$$

where each term is a matrix differential operator, the first one being the Boussinesq Poisson tensor:

$$Q_{-4,\boldsymbol{u}} = Q_{\boldsymbol{u}}^{\mathrm{Bou}}. \tag{4.8}$$

Now, let us consider the recombined vector field $Z_s^\epsilon$; from the Hamiltonian formulation (3.23), from the previous expansion of $Q^\epsilon$ and equation (4.6), we infer that

$$Z_s^\epsilon(\boldsymbol{u}) = \sum_{m=0}^{\infty} \epsilon^m Z_{s,m}(\boldsymbol{u}) \tag{4.9}$$

where $Z_{s,m}(\boldsymbol{u})$ depends polynomially on $\boldsymbol{u}$ and its derivatives. All these expansions will be used in the rest of the proof.

*Step 2. Refinement in the expansions of $B^\epsilon_s$ and $Z^\epsilon_s$.* From now on, we denote by $\mathcal{L}_W$ the Lie derivative along any vector field $W$; with this notation, the Lax formulation 3.23 of $Z^\epsilon_s$ is written as

$$\mathcal{L}_{Z^\epsilon_s} L^\epsilon = [B^\epsilon_s, L^\epsilon]. \tag{4.10}$$

We insert in this equation the expansions (4.9), (4.2) and (4.1), and infer that

$$Z_{s,m}(\boldsymbol{u}) = 0 \qquad \text{for } m < q. \tag{4.11}$$

This is obvious if $q = 0$; if $q > 0$ (which is indeed the case, as we will see later), (4.11) follows from comparison between the two sides of (4.10). In principle, the left-hand side contains all powers of $\epsilon$ of exponent greater than or equal to three; on the other hand, the right-hand side is clearly $\mathrm{O}(\epsilon^{q+3})$, so the coefficients of $\epsilon^3, \epsilon^4, \ldots, \epsilon^{q+2}$ in the left-hand side must be zero, yielding the equations

$$\mathcal{L}_{Z_{s,0}} L_3 = 0 \tag{4.12}$$

$$\mathcal{L}_{Z_{s,0}} L_4 + \mathcal{L}_{Z_{s,1}} L_3 = 0 \tag{4.13}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\mathcal{L}_{Z_{s,0}} L_{q+2} + \mathcal{L}_{Z_{s,1}} L_{q+1} + \cdots + \mathcal{L}_{Z_{s,q-1}} L_3 = 0. \tag{4.14}$$

From the explicit expression of $L_3 = \frac{4}{3} L^{\mathrm{Bou}}$ it is evident that $\mathcal{L}_W L_3 = 0$ for a vector field $W$ iff $W$ is identically zero. So, equation (4.12) implies $Z_{s,0} = 0$; inserting this result in (4.13) we obtain $\mathcal{L}_{Z_{s,1}} L_3 = 0$, which implies in turn $Z_{s,1} = 0$; iteration of this argument yields (4.11).

Now, let us return to equation (4.10), and equate the coefficients of $\epsilon^{q+3}$ in the two sides; the operator $L_3 = \frac{4}{3} L^{\mathrm{Bou}}$ is still involved, and we obtain

$$\mathcal{L}_{Z_{s,q}} L^{\mathrm{Bou}} = [B_{s,q}, L^{\mathrm{Bou}}]. \tag{4.15}$$

This means that, at each point $\boldsymbol{u} \in \mathcal{U}$, the differential operator $B_{s,q}(\boldsymbol{u})$ is compatible with the Boussinesq Lax operator (i.e. that its commutator with $L^{\mathrm{Bou}}(\boldsymbol{u})$ is tangent to the Lax submanifold in the space of differential operators). From here, and from a known result of Drinfeld and Sokolov [2], it follows that $B_{s,q}(\boldsymbol{u})$ is a linear combination of the form

$$B_{s,q}(\boldsymbol{u}) = \sum_i \gamma_{s,i} (L^{\mathrm{Bou}})^{i/3}_+(\boldsymbol{u}) \tag{4.16}$$

where $i$ is an integer index, running over a finite set of values; in principle, the real coefficients $\gamma_{s,i}$ could be functionals of the variables $\boldsymbol{u}$, but the features of the $B^\epsilon_s$ expansion ensure that they are constant†; we observe that it cannot be $\gamma_{s,i} = 0$ for each $i$, for this would contradict the assumption $B_{s,q} \neq 0$. Let us apply equation (4.16) at $\boldsymbol{u} = (0, 0)$; in this way we obtain $B_{s,q}(0, 0) = \sum_i \gamma_{s,i} \partial^i$, so that we can write

$$B^\epsilon_s(0, 0) = \epsilon^q \sum_i \gamma_{s,i} \partial^i + \mathrm{O}(\epsilon^{q+1}). \tag{4.17}$$

Comparing this equation with (3.25) (expressing the first $C^{(2)}_s$ condition), we infer

$$q = [3s/2] - 2 \tag{4.18}$$

$$\gamma_{s,i} = \begin{cases} 3^{[s/2]-1} & \text{if } i = [3s/2] - 2 \\ 0 & \text{otherwise.} \end{cases} \tag{4.19}$$

† Otherwise, the coefficients of the differential operator $B_{s,q}(\boldsymbol{u})$ would not be differential polynomials in $\boldsymbol{u}$.

From here and from equation (4.16) it follows that

$$B_{s,q}(\boldsymbol{u}) = 3^{[\frac{1}{2}s]-1}(L^{\mathrm{Bou}})_+^{\frac{1}{3}[\frac{3}{2}s]-\frac{2}{3}}(\boldsymbol{u}) = B_s^{\mathrm{Bou}}(\boldsymbol{u}) \tag{4.20}$$

at any $\boldsymbol{u} \in \mathcal{U}$. Inserting this result into equation (4.15), we also obtain

$$Z_{s,q}(\boldsymbol{u}) = Z_s^{\mathrm{Bou}}(\boldsymbol{u}). \tag{4.21}$$

Equations (4.18), (4.20) and (4.21) yield statements (3.29)–(3.30) in proposition 3.3. To conclude the proof, we must derive (3.31), which will be attained in the following.

*Step 3. $\epsilon$-expansion of $h_s^\epsilon$.* Let us insert into equation (3.24) the expansions (4.7), (4.6) and (4.9); recalling that $Z_{s,m} = 0$ for $0 \leqslant m \leqslant q-1$, we infer that the coefficients of $\epsilon^m$ on the right-hand side must also be zero for $0 \leqslant m \leqslant q-1$, yielding the equations

$$Q_{-4,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,4} = 0 \tag{4.22}$$

$$Q_{-4,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,5} + Q_{-3,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,4} = 0 \tag{4.23}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$Q_{-4,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,q+3} + Q_{-3,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,q+2} + \cdots + Q_{q-5,\boldsymbol{u}} d_{\boldsymbol{u}} h_{s,4} = 0. \tag{4.24}$$

Equation (4.22) tells us that $h_{s,4}$ is a Casimir of the Poisson tensor $Q^{\mathrm{Bou}} = Q_{-4}$; on the other hand, the only Casimirs of the Boussinesq theory which are ($x$-integrals of) differential polynomial densities in $\boldsymbol{u}$ are (up to constants) linear combinations with constant coefficients of $h_0^{\mathrm{Bou}}$ and $h_1^{\mathrm{Bou}}$, so we can write

$$h_{s,4}(\boldsymbol{u}) = \eta_s + \chi_s \int \mathrm{d}x\, u + \kappa_s \int \mathrm{d}x\, v. \tag{4.25}$$

We claim that the coefficients $\eta_s$, $\chi_s$ and $\kappa_s$ are zero; indeed by examination of the transformation (3.2) one infers that neither constants nor terms proportional to $\int \mathrm{d}x\, u$ or $\int \mathrm{d}x\, v$ can appear in the expansion of $h_s^\epsilon$†. In conclusion, we have $h_{s,4} = 0$; inserting this result into equation (4.23) we obtain $Q_{-4} d h_{s,5} = Q^{\mathrm{Bou}} d h_{s,5} = 0$, and this again implies $h_{s,5} = 0$. Iterating this argument, we infer

$$h_{s,m} = 0 \qquad \text{for } m \leqslant q+3. \tag{4.26}$$

Now, let us return to equation (3.24) and equate the coefficients of $\epsilon^q$ in the expansions of the two sides; since $Z_{s,q} = Z_s^{\mathrm{Bou}}$ and $Q_{-4} = Q^{\mathrm{Bou}}$, we obtain

$$Z_s^{\mathrm{Bou}}(\boldsymbol{u}) = Q_{\boldsymbol{u}}^{\mathrm{Bou}} d_{\boldsymbol{u}} h_{s,q+4} \tag{4.27}$$

which implies

$$h_{s,q+4} = h_s^{\mathrm{Bou}} \tag{4.28}$$

up to the addition of a Casimir; the same argument employed above shows that this additive term is necessarily zero. The last equation (with the result (4.18) for $q$) leads to (3.31) of proposition 3.3, whose proof is now concluded. □

† A constant term could appear at most at order zero in $\epsilon$; $\int \mathrm{d}x\, u$ and $\int \mathrm{d}x\, v$ could appear at most at orders two and three, respectively. On the other hand, we know that $h_{s,0} = h_{s,2} = h_{s,3} = 0$.

## 5. A sketch of the continuous limit for the $K_N$ system

The Lax operator for the $K_N$ system with any number $N$ of fields $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_N)$ is written in the introduction, see equation (1.3). The vector fields for this system have the Lax formulations $\mathrm{d}L^\epsilon/\mathrm{d}t_s = [A_s^\epsilon, L^\epsilon]$, where $A_s^\epsilon(\boldsymbol{\alpha}) := (1/(N+1))(L^\epsilon)_+^{2s}(\boldsymbol{\alpha})$ for $s = 0, 1, 2, \ldots$. With our notation, the $K_N$ Poisson tensor at any point $\boldsymbol{\alpha}$ sends an $N$-tuple $\delta\boldsymbol{\alpha} = (\delta\alpha_k)_{k=1,\ldots,N}$ into $\dot{\boldsymbol{\alpha}} = (\dot{\alpha}_j)_{j=1,\ldots,N}$, where

$$\dot{\alpha}_j = \frac{1}{N+1} \sum_{j=1}^{N} Q_{jk}^\epsilon \delta\alpha_k \tag{5.1}$$

$$Q_{jk}^\epsilon := \alpha_{k+j[2k\epsilon]}\Delta_{2k\epsilon} - \alpha_{k+j}\Delta_{-2j\epsilon} + \left(\sum_{l=0}^{2k-1} - \sum_{l=1-2j}^{2k-2j}\right)\alpha_{k[l\epsilon]}\alpha_j\Delta_{l\epsilon}$$
$$+ \left(\sum_{l=1}^{k-1} - \sum_{l=k-N}^{k-j-1}\right)\alpha_{k-l}\alpha_{j+l[2l\epsilon]}\Delta_{2l\epsilon}. \tag{5.2}$$

In the above sums, it is intended that $\alpha_i := 0$ if $i < 1$ or $i > N$ and $\sum_{l=a}^{b} := 0$ if $a > b$.

Each vector field of the $K_N$ hierarchy is Hamiltonian with respect to $Q^\epsilon$, the Hamiltonians being the traces of even powers of $L^\epsilon$.

It was shown in [5] that, under an appropriate field rescaling, the leading term in the $\epsilon$ expansion of $L^\epsilon(\boldsymbol{\alpha})$ is the $sl(N+1)$ KdV-type Lax operator. With our notation, the rescaling is the map $\boldsymbol{u} \mapsto \boldsymbol{\alpha} = \Theta^\epsilon(\boldsymbol{u})$ given by

$$\alpha_k = \frac{(-1)^{k-1}}{(2k-1)}\binom{N}{k} + \frac{(-1)^{k-1}}{2N+2}\sum_{m=2}^{k+1}(-2)^m\binom{N+1-m}{k+1-m}\epsilon^m u_{m-1}. \tag{5.3}$$

For $N = 1$, $\alpha_1 := \alpha$, $u_1 := u$ and for $N = 2$, $\alpha_1 := \alpha, \alpha_2 := \rho$, $u_1 := v$, $u_2 := u$ this transformation gives, respectively, the mappings (2.1) and (3.2). For $N$ arbitrary, the analysis of [5] shows that

$$L^\epsilon(\boldsymbol{\alpha})|_{\alpha=\Theta^\epsilon(\boldsymbol{u})} = \left(1 + \sum_{l=1}^{N}\frac{(-1)^{l-1}}{2l-1}\binom{N}{l}\right) + \frac{2^{N+1}}{2N+2}\epsilon^{N+1}L(\boldsymbol{u}) + \mathrm{O}(\epsilon^{N+2}) \tag{5.4}$$

where $L(\boldsymbol{u})$ is the Lax operator of the $sl(N+1)$ KdV theory (see (1.4)). Combining this result with the method of sections 2 and 3, one could show that appropriate recombinations of the companion operators $A_s^\epsilon$ and of the $K_N$ vector fields give, in the $\epsilon \mapsto 0$ limit, the homologous objects of the $sl(N+1)$ KdV theory.

The Hamiltonian formalism of this theory (with a recombination scheme for the Hamiltonians) could also be reconstructed, provided that one proves the following: if the transformation (5.3) is applied to the Kupershmidt Poisson tensor (5.2), the leading term in the $\epsilon$ expansion is the first $sl(N+1)$ Poisson structure. In principle, the verification of this statement could be performed directly, using the explicit expressions for the $K_N$ Poisson tensor and the field rescaling. As a matter of fact, the computation is very difficult for $N$ arbitrary, so we have limited the verification to the case $N = 3$.

Let us introduce the notations $\alpha_1 := \alpha$, $\alpha_2 := \rho$, $\alpha_3 := \sigma$. The $K_3$ Poisson tensor has the matrix representation

$$Q_\alpha^\epsilon = \frac{1}{4}\begin{pmatrix} Q_{\alpha\alpha}^\epsilon & Q_{\alpha\rho}^\epsilon & Q_{\alpha\sigma}^\epsilon \\ Q_{\rho\alpha}^\epsilon & Q_{\rho\rho}^\epsilon & Q_{\rho\sigma}^\epsilon \\ Q_{\sigma\alpha}^\epsilon & Q_{\sigma\rho}^\epsilon & Q_{\sigma\sigma}^\epsilon \end{pmatrix} \tag{5.5}$$

$$
\begin{aligned}
Q^{\epsilon}_{\alpha\alpha} &:= \rho_{[2\epsilon]}\Delta_{2\epsilon} + \alpha\alpha_{[\epsilon]}\Delta_{\epsilon} - \alpha\alpha_{[-\epsilon]}\Delta_{-\epsilon} - \rho\Delta_{-2\epsilon}\\
Q^{\epsilon}_{\alpha\rho} &:= \sigma_{[4\epsilon]}\Delta_{4\epsilon} + \alpha\rho_{[3\epsilon]}\Delta_{3\epsilon} + \alpha\rho_{[2\epsilon]}\Delta_{2\epsilon} - \alpha\rho - \alpha\rho_{[-\epsilon]}\Delta_{-\epsilon} - \sigma\Delta_{-2\epsilon}\\
Q^{\epsilon}_{\alpha\sigma} &:= \alpha\sigma_{[5\epsilon]}\Delta_{5\epsilon} + \alpha\sigma_{[4\epsilon]}\Delta_{4\epsilon} - \alpha\sigma - \alpha\sigma_{[-\epsilon]}\Delta_{-\epsilon}\\
Q^{\epsilon}_{\rho\alpha} &:= \sigma_{[2\epsilon]}\Delta_{2\epsilon} + \alpha_{[\epsilon]}\rho\Delta_{\epsilon} + \alpha\rho - \alpha_{[-2\epsilon]}\rho\Delta_{-2\epsilon} - \alpha_{[-3\epsilon]}\rho\Delta_{-3\epsilon} - \sigma\Delta_{-4\epsilon}\\
Q^{\epsilon}_{\rho\rho} &:= \rho_{[3\epsilon]}\rho\Delta_{3\epsilon} + (\alpha\sigma_{[2\epsilon]} + \rho_{[2\epsilon]}\rho)\Delta_{2\epsilon} + \rho_{[\epsilon]}\rho\Delta_{\epsilon} - \rho_{[-\epsilon]}\rho\Delta_{-\epsilon}\\
&\quad -(\alpha_{[-2\epsilon]}\sigma + \rho_{[-2\epsilon]}\rho)\Delta_{-2\epsilon} - \rho_{[-3\epsilon]}\rho\Delta_{-3\epsilon}\\
Q^{\epsilon}_{\rho\sigma} &:= \rho\sigma_{[5\epsilon]}\Delta_{5\epsilon} + \rho\sigma_{[4\epsilon]}\Delta_{4\epsilon} + \rho\sigma_{[3\epsilon]}\Delta_{3\epsilon} + \rho\sigma_{[2\epsilon]}\Delta_{2\epsilon} - \rho\sigma - \rho\sigma_{[-\epsilon]}\Delta_{-\epsilon}\\
&\quad -\rho\sigma_{[-2\epsilon]}\Delta_{-2\epsilon} - \rho\sigma_{[-3\epsilon]}\Delta_{-3\epsilon}\\
Q^{\epsilon}_{\sigma\alpha} &:= \alpha_{[\epsilon]}\sigma\Delta_{\epsilon} + \alpha\sigma - \alpha_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \alpha_{[-5\epsilon]}\sigma\Delta_{-5\epsilon}\\
Q^{\epsilon}_{\sigma\rho} &:= \rho_{[3\epsilon]}\sigma\Delta_{3\epsilon} + \rho_{[2\epsilon]}\sigma\Delta_{2\epsilon} + \rho_{[\epsilon]}\sigma\Delta_{\epsilon} + \rho\sigma - \rho_{[-2\epsilon]}\sigma\Delta_{-2\epsilon} - \rho_{[-3\epsilon]}\sigma\Delta_{-3\epsilon}\\
&\quad -\rho_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \rho_{[-5\epsilon]}\sigma\Delta_{-5\epsilon}\\
Q^{\epsilon}_{\sigma\sigma} &:= \sigma_{[5\epsilon]}\sigma\Delta_{5\epsilon} + \sigma\sigma_{[4\epsilon]}\Delta_{4\epsilon} + \sigma\sigma_{[3\epsilon]}\Delta_{3\epsilon} + \sigma_{[2\epsilon]}\sigma\Delta_{2\epsilon} + \sigma\sigma_{[\epsilon]}\Delta_{\epsilon} - \sigma_{[-\epsilon]}\sigma\Delta_{-\epsilon}\\
&\quad -\sigma_{[-2\epsilon]}\sigma\Delta_{-2\epsilon} - \sigma_{[-3\epsilon]}\sigma\Delta_{-3\epsilon} - \sigma_{[-4\epsilon]}\sigma\Delta_{-4\epsilon} - \sigma_{[-5\epsilon]}\sigma\Delta_{-5\epsilon}.
\end{aligned}
$$

Setting $u_1 := u$, $u_2 := v$, $u_3 := w$, the transformation (5.3) takes the form

$$
\begin{aligned}
\alpha &= 3 + \tfrac{1}{2}\epsilon^2 u\\
\rho &= -1 - \epsilon^2 u + \epsilon^3 v\\
\sigma &= \tfrac{1}{5} + \tfrac{1}{2}\epsilon^2 u - \epsilon^3 v + 2\epsilon^4 w.
\end{aligned}
\tag{5.6}
$$

The pullback of the Poisson tensor (5.5) along this transformation is described again by equation (3.5). Explicitly, at each point $\boldsymbol{u} = (u, v, w)$ we have

$$
\begin{aligned}
Q^{\epsilon}_{\boldsymbol{u}} = &\begin{pmatrix} 2/\epsilon^2 & 0 & 0\\ 2/\epsilon^3 & 1/\epsilon^3 & 0\\ 1/2\epsilon^4 & 1/2\epsilon^4 & 1/2\epsilon^4 \end{pmatrix}
\begin{pmatrix} \frac14 Q^{\epsilon}_{\alpha\alpha} & \frac14 Q^{\epsilon}_{\alpha\rho} & \frac14 Q^{\epsilon}_{\alpha\sigma}\\ \frac14 Q^{\epsilon}_{\rho\alpha} & \frac14 Q^{\epsilon}_{\rho\rho} & \frac14 Q^{\epsilon}_{\rho\sigma}\\ \frac14 Q^{\epsilon}_{\sigma\alpha} & \frac14 Q^{\epsilon}_{\sigma\rho} & \frac14 Q^{\epsilon}_{\sigma\sigma} \end{pmatrix}\\
&\times \begin{pmatrix} 2/\epsilon^2 & 2/\epsilon^3 & 1/2\epsilon^4\\ 0 & 1/\epsilon^3 & 1/2\epsilon^4\\ 0 & 0 & 1/2\epsilon^4 \end{pmatrix}.
\end{aligned}
\tag{5.7}
$$

Expanding in $\epsilon$, we have found

$$
Q^{\epsilon}_{\boldsymbol{u}} = \frac{1}{5\epsilon^5}\begin{pmatrix} 0 & 0 & 4\partial_x\\ 0 & 4\partial_x & 2\partial_{xx}\\ 4\partial_x & -2\partial_{xx} & 2\partial_{xxx} + 2u\partial_x + u_x \end{pmatrix} + \mathrm{O}\left(\frac{1}{\epsilon^4}\right). \tag{5.8}
$$

So the continuous limit of the $K_3$ Poisson structure has the expected behaviour: the coefficient of $(1/\epsilon^5)$ is recognized to be the first Poisson tensor of the $sl(4)$ KdV theory.

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